Metric-Field Conditioning for Mesh Adaptation Problems

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Summary. Metric tensor fields computed from solution data for adaptation problems are not always well-behaved. They may contain noise, high gradients in desired directionality and anisotropy, global undulations in directionality, and other defects that pose challenges for meshing algorithms. We focus on the application of filters to address some of these defects through noise removal and a form of fairing. At the same time, it is desirable to retain characteristics of the original metric data. These methods are demonstrated on metric data from a high-speed, 2D flow problem.

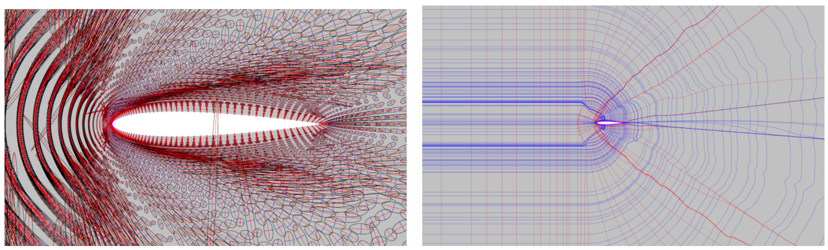
1 Introduction

Mesh adaptation is a process that can be employed in domains such as computational fluid dynamics to improve solution accuracy, convergence, and resolution of various physical phenomena. Adaptation involves local mesh modification [1] or complete remeshing of the domain according to *a posteriori* requirements computed from solution-based error estimates. Such requirements consist of preferred mesh directionality and anisotropy, and are commonly represented as Riemannian metric tensors. There are many formulations that produce metric tensors from a solution field; here we consider gradient-based and adjoint-based methods that rely on Hessian information from a scalar solution variable to determine directionality and aspect ratio, which is then combined with a scalar intensity that is specific to each method [2, 1].

Partially due to the numerical approximation involved, solution-based metrics from these methods are not well-behaved. Simple observations of the resulting metrics (refer to Figure 1) reveal two types of issues:

- Isolated, local noise: individual nodal metrics may be outliers in terms of directionality and/or anisotropy when compared to their local neighborhood. These also lead to a high anisotropy or directionality gradient when interpolating the metrics. Several anisotropy outliers can be seen in Figure 1a. The method of visualization is a combination of tensor ellipses and hedgehogs (arrows) that indicate the local directionality and sizes along those directions. Directions corresponding to the larger size of the two are red, the others are blue.

- Global undulations in directionality: as seen in Figure 1b, metric streamlines in proximity to a bow shock for a symmetric airfoil problem are wavy. Metric streamlines are similar to flow streamlines, but instead follow the directions (eigenvectors of the metric tensor) corresponding to either the larger or smaller size. A consistent color mapping scheme is employed.



(a) Ellipses (zoomed in) (b) Metric streamlines

Fig. 1. Noise and undulations in solution-based metric data

These two types of issues can cause difficulties for meshers in terms of quality, computational effort, and suitability of the resulting mesh for analysis. Therefore, we seek methods that can remove aspect-ratio, scaling, and directionality outliers, preserve strong features in the metric field, and fair directionality trends.

These methods are applied in the context of a metric-streamline based mesher (that is also used for visualization purposes) [3, 4] as well as a packing-based approach [5].

2 Metric Conditioning

Median filtering is a traditional image processing technique that is effective at removing noise while preserving important image features. When applied to a scalar grayscale image, for example, one replaces a grayscale pixel with the median value of the pixels in its neighborhood. In [6], Welk et al. extend the definition of a median to tensors and present two variants of the median filter for tensor-valued images.

The typical algorithm for computing a median of a set of scalar values is to sort them by value and then select the middle value. It is not as straightforward to sort SPD tensor data in this manner, so Welk et al. consider an alternative definition: the median element of a set of values is the value that has a minimum summed distance to the other elements. This can be stated as the following for a set of values $\mathbb{X} = \{x_1, \ldots, x_N\}$:

$$x_{\text{median}} := \underset{x \in \mathbb{X}}{\arg\min} \sum_{j=1}^{N} ||x - x_j|| \ , \tag{1}$$

where x_{median} is the median value, x is restricted to the set $\mathbb{X}$ of elements x_j. For scalar values, the absolute value is the same as the L^1 norm and is consistent with the sorting method for the median.

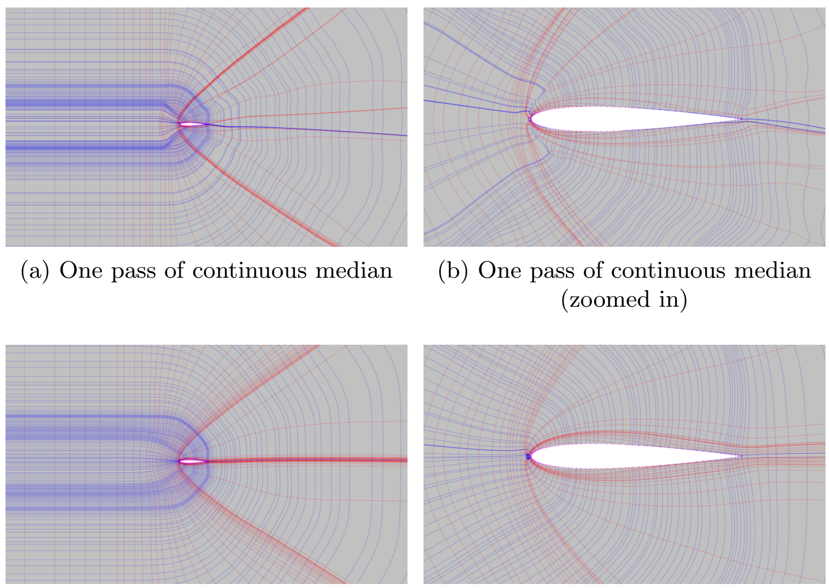
We now discuss specific choices for the filtering of solution-based metrics. Out of several possible norms, the Frobenius, or Euclidean, norm has been selected for the metric norm in Equation (1). It has the ideal property of being a tensor invariant. Unlike diffusion tensor data, metric data is non-uniformly sampled over space. To compensate for this, the local neighborhood around a node can be built by including all neighbors within a certain radius. Neighborhood construction has been designed to satisfy criteria such as minimum number of layers (n-ring), a neighborhood radius, and a minimum number of neighbors. Furthermore, a Gaussian distance weighting scheme is applied to the terms of the summation in Equation (1) to smoothly factor in distance between the metric samples in a neighborhood. This yields the following median around a metric for a set of metrics $\mathbb{M} = \{\mathcal{M}_1, \ldots, \mathcal{M}_N\}$ at corresponding positions $\mathbb{X} = \{x_1, \ldots, x_N\}$:

$$\begin{aligned} \mathcal{M}_{\text{median}} &:= \underset{\mathfrak{m} \in \mathbb{M}}{\arg\min} \left[\left(\sum_{i}^{N} \omega_i \right)^{-1} \sum_{i}^{N} \omega_i \, ||\mathfrak{m} - \mathcal{M}_i||_2 \right] \\ \omega_i &= \exp\left(- \frac{||x - x_i||_2^2}{2\sigma^2} \right) \\ \sigma^2 &= \frac{1}{N} \sum_{i}^{N} ||x - x_i||_2^2 \, , \end{aligned} \tag{2}$$

where $\mathfrak{m}$ is a dummy variable, and the Gaussian weights ω_i bias the computation with respect to a metric at position x. This is also reflected in the variance, σ^2. Technically, the sets $\mathbb{M}$ and $\mathbb{X}$ may contain the metric that is current being filtered, i.e., $\mathcal{M}$ at x. A continuous extension is also used, where the restriction that $\mathfrak{m} \in \mathbb{M}$ is relaxed; this allows the median to take any symmetric positive-definite value (new restriction: $\mathfrak{m} \in \mathbb{S}^{+}_{2\times 2}$). In the present work, this is solved using gradient descent with adaptive step size control and the weighted mean of the metrics in $\mathbb{M}$ as an initial guess.

3 Results

The discrete and continuous median filters are applied using Jacobi-style iterations over a specified number of global passes over the domain. Figure 2 shows the effects on the metric streamlines after one and ten passes of median filtering on the raw metrics from Figure 1 for a 1-ring neighborhood. In practice, filtering is applied conservatively (one pass of discrete median) to produce a rectified field for length measurements. Then, a more generous application (ten passes of continuous median filtering) is used to generate a field with faired directionality. A family of filtered fields can be produced in this manner for consumption by a meshing algorithm. Finally, Figure 3 provides a global perspective of the effects on directionality.



(a) One pass of continuous median

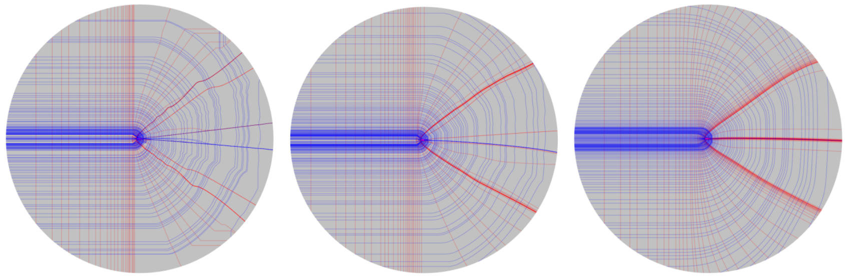
(b) One pass of continuous median (zoomed in)

(c) Ten passes of continuous median

(d) Ten passes of continuous median (zoomed in)

Fig. 2. Gaussian weighted median filtering of solution-based metric fields

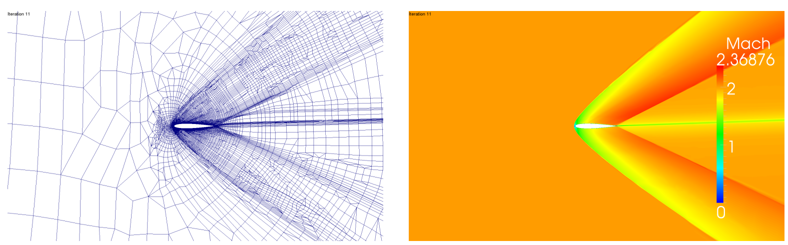
We apply the methods described in the present work to a supersonic, steady flow over a 0012 airfoil. The airfoil has a reference length of 1, and is embedded in a circular domain with a radius of 20 chord-lengths. The parameters specified for the laminar, viscous analysis are as follows: $Mach = 2$, $Re = 1 \times 10^6$, and the angle of attack, $\alpha = 2°$. Using the FUN3D solver [7], the solutions are performed according a schedule that starts off with first-order accurate iterations, followed by second-order accurate iterations during which a flux limiter is enabled. The inviscid flux construction scheme is `LDFSS`, and the `hVanAlbada` flux limiter is used. The initial mesh for these adaptation iterations was prepared by running eight adaptation iterations with proposed method on an initial unadapted quad mesh with $Re = 1 \times 10^3$. Filtered metrics



(a) Raw metrics (b) One pass continuous median (c) Ten passes continuous median

Fig. 3. Global view of metric streamlines after filtering

are passed to a streamline and rectangular cell packing mesher that produces the mesh and solution (Mach plot) in Figure 4, demonstrating anisotropic capture of the shock and wake phenomena without the waviness observed in Figure 1b and without washing away or distorting strong features as is possible with simple smoothing.



(a) Iteration 11 mesh (b) Iteration 11 solution

Fig. 4. Adapted meshes and Mach plots for the 0012 airfoil (Iteration 11)

4 Conclusion

A method for conditioning solution-based metric tensor data for adaptation problems was presented. Discrete and continuous variants of a tensor-valued Median filter were used to denoise and fair a metric field while preserving some characteristics of the original metric field. The method was applied to data from a viscous, supersonic flow problem and applied to an adaptation process to capture strong flow features.

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